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Forced Vibration Analysis of Axially FG Straight Beams by Mixed FEM

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ABSTRACT

The objective of this study is to investigate the forced vibration analysis of axially functionally graded (FG) straight beams having square and rectangular cross-sections. For this purpose, a mixed finite element formulation based on Timoshenko beam assumption is implemented. Two-noded straight element is used to discretize beam domain. The numerical analyses are performed for dynamic rectangular impulsive type of uniformly distributed load in the Laplace space, next the results are transformed back to time space numerically using the modified Durbin's algorithm. A parametric study is performed to investigate the influence of the material gradients on the dynamic analysis of axially FG beams having different cross-sections (square and rectangular) and boundary conditions (fixed-fixed, fixed-pinned).

Keywords: axially FG material, finite element, forced vibration, straight beam, Timoshenko beam theory

1. INTRODUCTION

Straight beams are widely used in several fields such as, mechanical, civil, aerospace engineering *etc.* with a growing demand in defense, transportation or aerospace structures they are designed for advanced materials where the required material properties can be selected regarding the specifications of the application. Composite or functionally graded materials are also preferred due to their strength, thermal characteristics, and lightness properties in several engineering practice. In the literature, there are many studies about free vibration analysis of axially FG straight beams. Some of these are cited as follows: (Wu *et al*., 2005; Huang *et al*., 2010; Alshorbagy *et al*., 2011; Akgöz and Civalek, 2013; Huang *et al*., 2013; Li *et al*., 2013; Rajasekaran and Norouzzadeh Tochaei, 2014; Sarkar and Ganguli, 2014; Zeighampour and Beni, 2015; Rezaiee-Pajand and Hozhabrossadati, 2016; Zhao *et al*., 2017; Cao *et al*., (2018, 2019); Zhou and Zhang, 2019). The dynamic analysis of FG beams are cited as follows: (Liang and Zhang, 2015) studied dynamic behavior of rotating axially (FG) tapered beams based on a new dynamic model utilizing the B-spline method. (Çalım, 2016a) investigated the transient analysis of axially FG straight Timoshenko beam with variable cross-section by using complementary functions method (CFM). (Çalım, 2016b) analyzed the free and forced vibration characteristics of axially FG Timoshenko beams resting on two-parameter viscoelastic foundation by adopting CFM. (Wang and Wu, 2016) examined dynamic responses of an axially FG beam under thermal environment subjected to a moving harmonic load. (Hao and Wei, 2016) evaluated the dynamic characteristics of bi-directionally FG Timoshenko beams. (Xie *et al*., 2018) concerned with the dynamic response of an axially FG beam with longitudinal-transverse coupling effect under a moving transverse/longitudinal harmonic load.

In this study, the forced vibration response of axially functionally graded (FG) straight beams is investigated under dynamic rectangular impulsive type of uniformly distributed load by implementing a mixed finite element method (MFEM). The solution of the structural problem is carried out in frequency domain by Laplace transformation and the results are transformed back to the time domain numerically by means of the Modified Durbin's transformation algorithm (Dubner and Abate, 1968; Durbin, 1974; Narayanan, 1980). The detailed explanation for mixed finite element formulation of space curve beam in Laplace domain exists in (Eratlı *et al*., 2014). The influence of material gradients on the dynamic behaviour of axially FG straight beam is treated over different cross-sections and boundary conditions. Axially FG material distribution is assumed as a power-law relation. The square and rectangular cross sections are chosen to have the same constant area. The fixed-fixed and fixed-pinned boundary conditions are considered. Verification of the solutions are carried on ANSYS.

2. FORMULATION

2.1. The functional in Laplace Domain

The field equations based on Timoshenko beam theory for the isotropic homogenous elastic spatial beam exist in (Aköz *et al*., 1991), and, (Omurtag and Aköz, 1992). Using Cartesian coordinate system, letting x be the axis of axially functionally graded straight beam and $\rho = \rho(x)$ is the material density, the functional is transformed to frequency domain by Laplace transformation for the dynamic analysis of the beam as follows:

$$\left.\begin{aligned}\mathbf{I} = & -\left[\bar{u}_z, \bar{T}_{z,x}\right] + \left[\bar{\Omega}_y, \bar{T}_z\right] - \left[\bar{M}_{y,x}, \bar{\Omega}_y\right] - \tfrac{1}{2}\frac{1}{EI_y}\left[\bar{M}_y, \bar{M}_y\right] \\ & - \tfrac{1}{2}\frac{k'}{GA}\left[\bar{T}_z, \bar{T}_z\right] + \tfrac{1}{2}\rho A s^2\left[\bar{u}_z, \bar{u}_z\right] + \tfrac{1}{2}\rho\, s^2 I_y\left[\bar{\Omega}_y, \bar{\Omega}_y\right] \\ & - \left[\bar{q}_z, \bar{u}_z\right] - \left[\bar{m}_y, \bar{\Omega}_y\right] + \left[\left(\bar{T}_z - \hat{\bar{T}}_z\right), \bar{u}_z\right]_\sigma + \left[\left(\bar{M}_y - \hat{\bar{M}}_y\right), \bar{u}_z\right]_\sigma \\ & + \left[\hat{\bar{u}}_z, \bar{T}_z\right]_\varepsilon + \left[\hat{\bar{\Omega}}_y, \bar{M}_y\right]_\varepsilon\end{aligned}\right\} \qquad (1)$$

s is the Laplace transformation parameter, and the Laplace transformed variables are denoted by the over bars. $\bar{u}$, $\bar{\Omega}$, $\bar{T}$ and $\bar{M}$ are the displacement, rotation, force and moment in Laplace space, respectively. A is the cross sectional area, k' is the shear correction factor, I_y is the moment of inertia, E is the modulus of elasticity, G is the shear modulus. $\bar{q}_z$ and $\bar{m}_y$ are the distributed external force and moment in Laplace space. The parentheses in Eq. (1) indicate the inner product, and the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively. The field equations and functional for elastic spatial bars exists in (Omurtag and Aköz, 1992) and for the viscoelastic material case they are given in (Eratlı *et al.*, 2014). In this study, the functional is adapted for elastic forced vibration analysis of axially FG straight beams.

2.2. The mixed finite element formulation

Two-noded straight finite element is formulated using linear shape functions. Nodal variables are vertical displacement, rotation of cross-section, transverse shear force, and bending moment. Explicit form of the mixed finite element matrices of spatial bar exists in (Omurtag and Aköz, 1992).

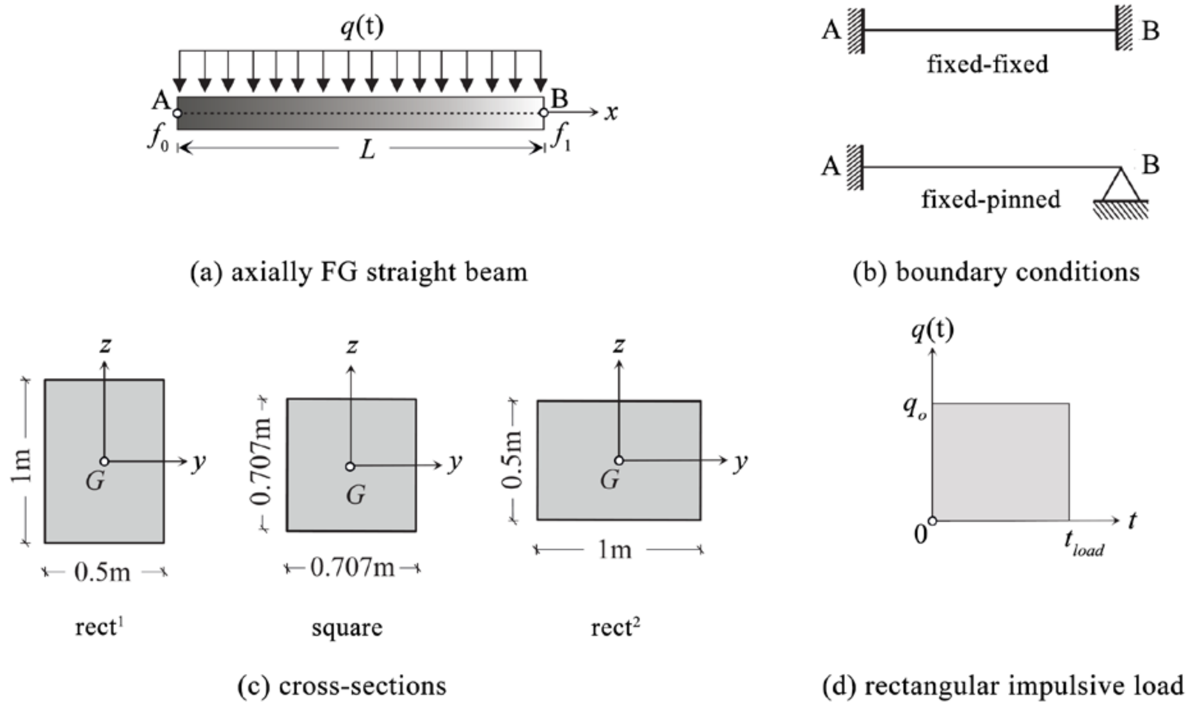

Figure 1. Axially FG straight beam, cross-sections and impulsive load.

2.3. Axially FG straight beam geometry

Axially FG material distribution is assumed as a power-law relation along the axis of straight beam expressed by:

$$f(x) = f_0 + (f_1 - f_0)\left(\frac{x}{L}\right)^m \tag{2}$$

where f denotes a material property (*e.g.* modulus of elasticity: E, density: ρ or modulus of shear: G), m is the material gradient index, the subscript "0" and "1" denotes the materials at left and right ends of the beam, respectively. L is the length of the beam (see Figure 1(a)).

3. RESULTS AND DISCUSSION

The forced vibration analysis of axially FG straight beam that is subjected to a dynamic rectangular impulsive type of uniformly distributed load $q = q(t)$ (see Figure 1(a)) is investigated. Through the analysis, different boundary conditions (fixed-fixed and fixed-pinned, Figure 1(b)), and different cross-sections (rectangular cross-sections and square cross-section, Figure 1(c)) are handled. The geometrical parameters of straight beam: the length of beam is $L = 5\,\text{m}$. The dimensions of the rectangular cross-sections for rect^1 and rect^2, and dimension of square cross-section are given in Figure 1(c). The net areas of the all three cross-sectional geometries (rect^1, rect^2, square) are equal to each other. The material properties are $E_0 = 210\,\text{GPa}$, $\rho_0 = 7500\,\text{kg/m}^3$, $E_1 = 70\,\text{GPa}$, $\rho_1 = 2500\,\text{kg/m}^3$ and the Poisson's ratio $\upsilon = 0.3$. The material gradients are $m = 0, 0.5, 1, 2, 3$ in Eq. (2). The magnitude and the duration of the load are $q_o = 200\,\text{kN/m}$ and $t_{load} = 0.021\text{s}$, respectively (see Figure 1(d)). The dynamic response of the beam is determined within $0 \le t \le 0.08\text{s}$. The parameters $N = 2^{11}$ and $aT = 6$ are used in analysis for inverse Laplace transformation algorithm (Eratlı *et al*., 2014).

Convergence Analysis: Through the analysis, the vertical displacement u_z and the rotation Ω_y at midpoint, and the shear force T_z and the moment M_y at point A (see Figure 1(a)) for an axially FG straight beam having fixed-fixed boundary condition and material gradient index $m = 3$ are obtained using 20, 30, 40 finite elements. It is observed that, the results of 30 and 40 elements coincide with each other (see Figure 2). Consequently, in the following examples, 40 elements are employed.

Verification: The axially FG straight beam having material index $m = 1$, a square cross-section and fixed-fixed boundary condition is considered and solved via MFEM and ANSYS for 40 elements. In ANSYS solution, the beam elements (BEAM188) are used and the axially FG material is defined by the average value of material properties between each node of the straight beam (41 nodes). The comparison of MFEM and ANSYS results are given in Figure 3 for the vertical displacement u_z at midpoint, and the force T_z and the moment M_y at point A of axially FG straight beam. The percent differences of the results obtained by ANSYS with respect to the results of MFEM are given in Figure 3 for some peak points. It is observed that the results of MFEM and ANSYS are quite in agreement with each other.

The effect of the material gradient index: The material axially gradient indexes are $m = 0, 0.5, 1, 2, 3$. The geometry of cross-sectional area of the beam is square (see Figure 1(c)). Two boundary conditions are chosen fixed-fixed and fixed-pinned supports (see Figure 1(b)). The time histories of the vertical displacement u_z and the rotation Ω_y at midpoint of the axially FG beams for fixed-fixed and fixed- pinned supports are given in Figures 4(a-b) and 5(a-b), respectively. The time histories of force T_z and the moment M_y at point A of the axially FG beams fixed at both ends are given in Figures 4(c-d), respectively. The values of first extrema of the forced vibration zone for the vertical displacement u_z and the rotation Ω_y at midpoint of the axially FG beam are examined. For this purpose, the u_z and Ω_y values of first extrema of axially FG beam having $m = 1, 2, 3$ are normalized with respect to the results of $m = 0.5$. The percent reductions are tabulated in Table 1 for the both boundary conditions.

The effect of the cross section: Three cross sectional geometries rect^1, rect^2 and a square are chosen, (see Figure 1(c)). Their net cross-sectional areas are equal to each other. The axially material gradient is $m = 1$. The boundary condition is fixed at both ends. The time histories of the vertical displacement u_z and the rotation Ω_y at midpoint of the axially FG beam are given in Figure 6. The values of first extrema of the forced vibration zone for the vertical displacement u_z and the rotation Ω_y at midpoint of the axially FG beam are examined. For this purpose, the u_z (see Figure 6(a)) and Ω_y (see Figure 6(b)) values of first extrema of axially FG beam having square and rect^1 cross section are normalized with respect to the results of rect^2 cross section. The percent reductions can be given as follows: in the case of the square cross section for u_z and Ω_y, they are 43.6% and 46.7%, respectively. In the case of the rect^1 cross section for u_z and Ω_y, they are 66.1% and 69.0%, respectively.

The effect of the boundary conditions: The boundary conditions are fixed-fixed and fixed-pinned (see Figure 1(b)). The material gradient is $m = 1$. The geometry of cross-sectional area is square. The values of first extrema of the vertical displacement u_z and the rotation Ω_y at midpoint of the axially

FG beam are examined for the forced vibration zone. For this purpose, u_z and Ω_y values of first extrema of the axially FG beam having fixed-fixed boundary condition (see Figures 4(a-b)) are normalized with respect to the results of fixed-pinned boundary condition (see Figures 5(a-b)). The percent reductions are 37.6% for u_z and 70.4% for Ω_y.

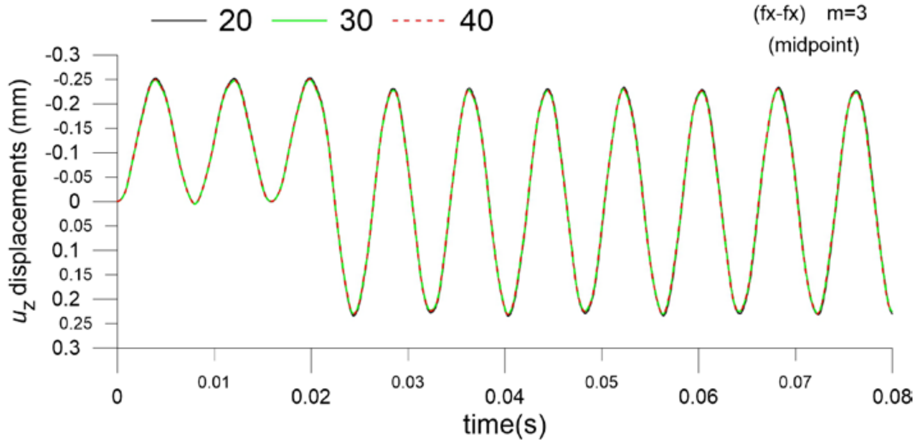

Figure 2(a). Convergence analysis of mixed FE for vertical displacement u_z at midpoint of axially FG beam having square cross section.

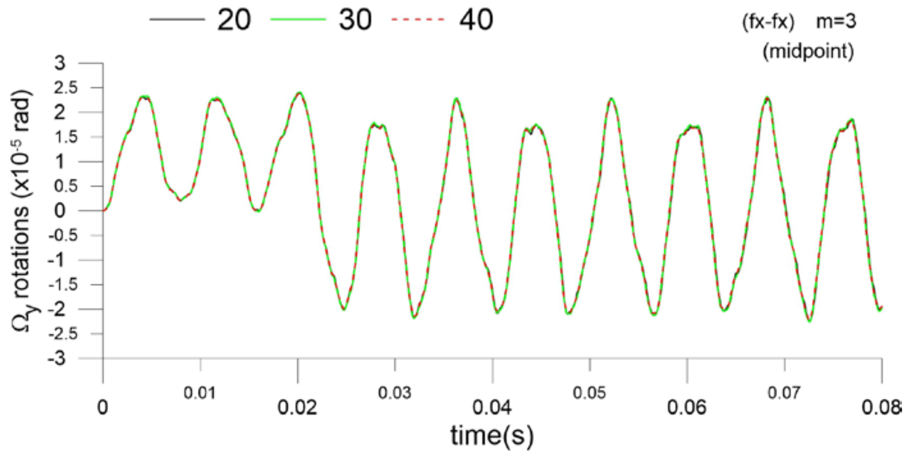

Figure 2(b). Convergence analysis of mixed FEM for rotation Ω_y at midpoint of axially FG beam having square cross section.

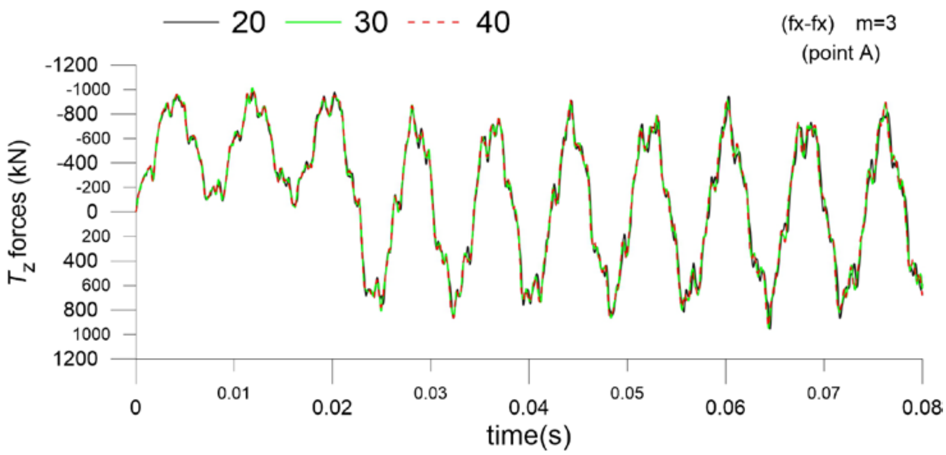

Figure 2(c). Convergence analysis of mixed FEM for forces T_z at point A of axially FG beam having square cross section.

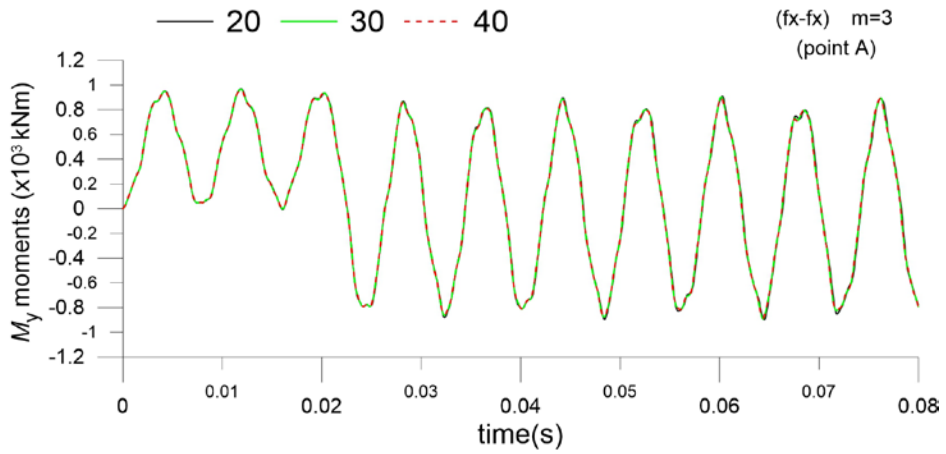

Figure 2(d). Convergence analysis of mixed FE for moment M_y at point A of axially FG beam having square cross section.

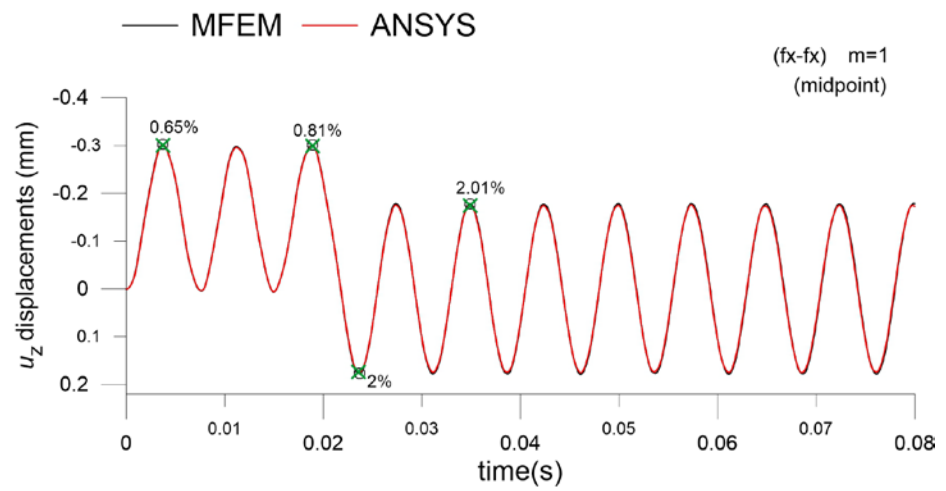

Figure 3(a). Comparison of MFEM and ANSYS for vertical displacement u_z at midpoint of axially FG beam having square cross section. (O: result of MFEM, ×: result of ANSYS)

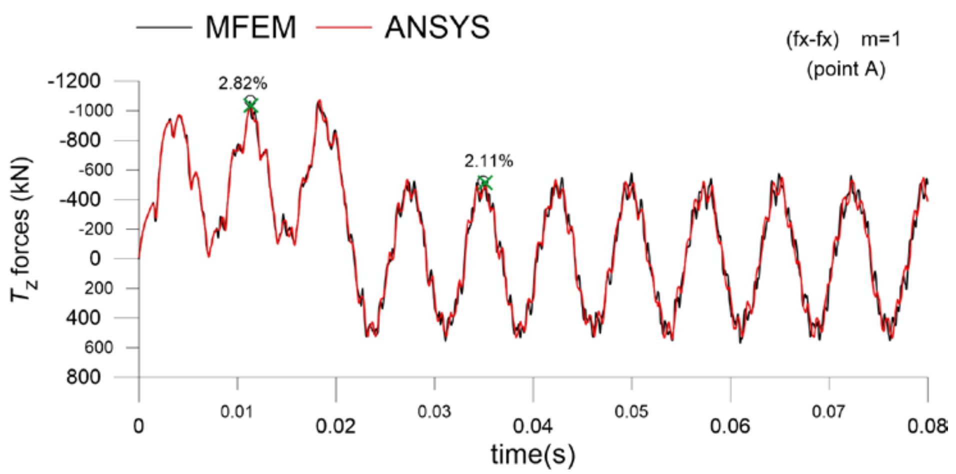

Figure 3(b). Comparison of MFEM and ANSYS for forces T_z at point A of axially FG beam having square cross section. (O: result of MFEM, ×: result of ANSYS)

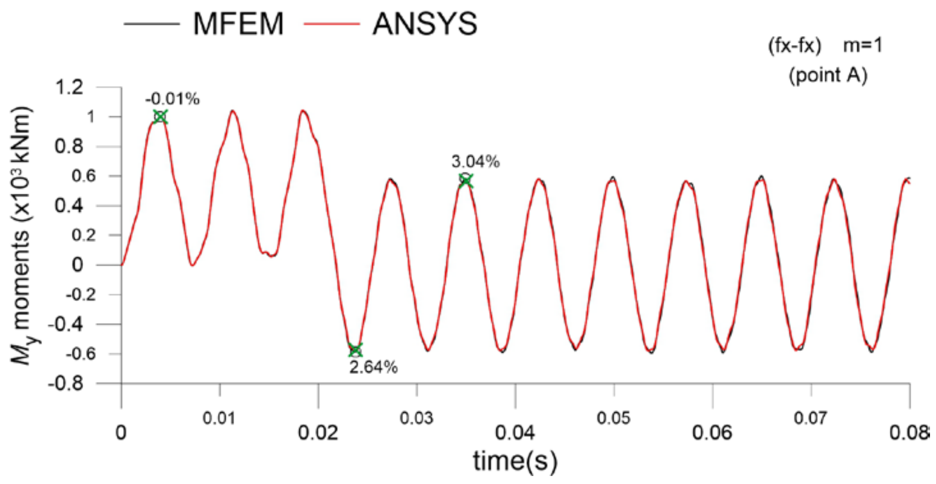

Figure 3(c). Comparison of MFEM and ANSYS for moment M_y at point A of axially FG beam having square cross section. (○: result of MFEM, ×: result of ANSYS)

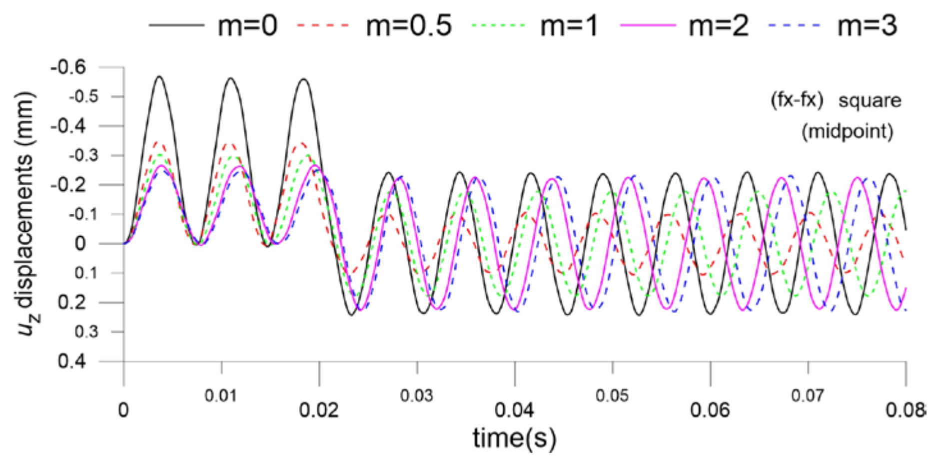

Figure 4(a). The vertical displacement u_z at midpoint of axially FG beam having square cross-section fixed-fixed (fx-fx) boundary condition for $m = 0, 0.5, 1, 2, 3$.

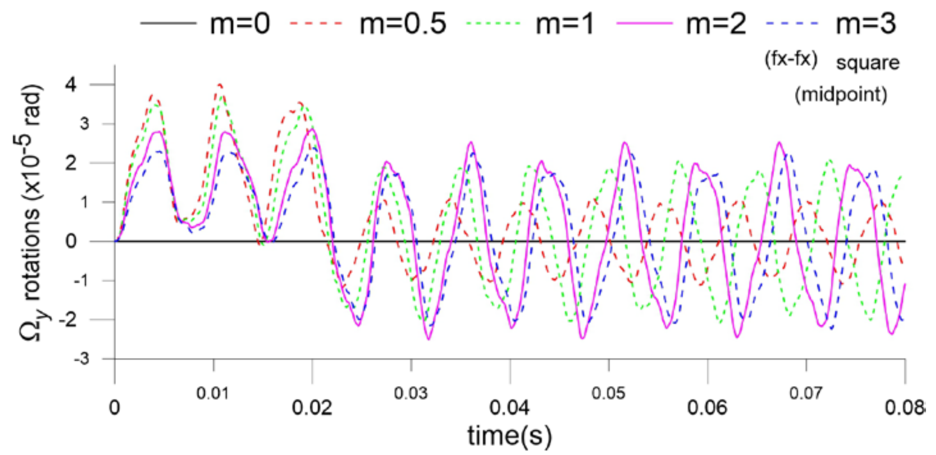

Figure 4(b). The rotation Ω_y at midpoint of axially FG beam having square cross-section fixed-fixed (fx-fx) boundary condition for $m = 0, 0.5, 1, 2, 3$.

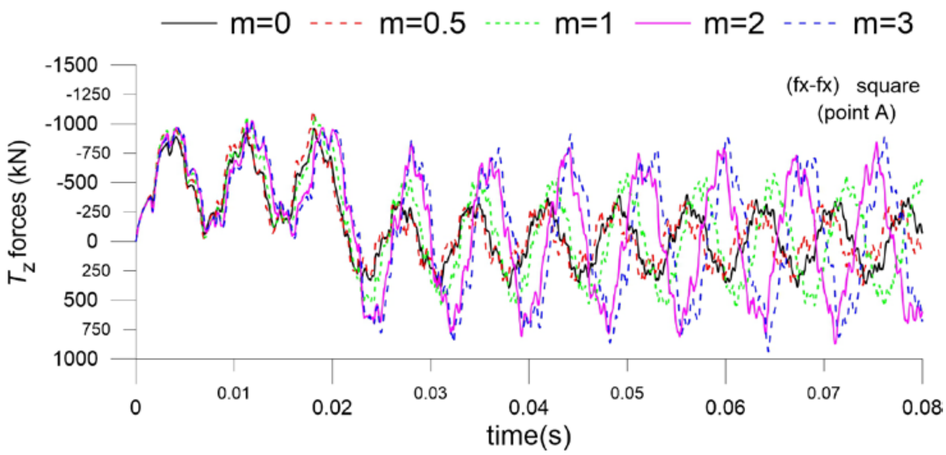

Figure 4(c). The forces T_z at point A of axially FG beam having square cross-section fixed-fixed (fx-fx) boundary condition for $m = 0, 0.5, 1, 2, 3$.

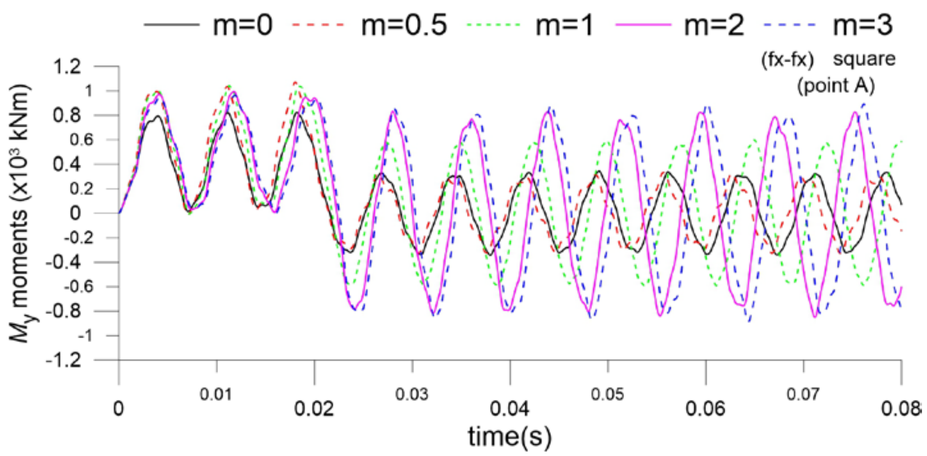

Figure 4(d). The moment M_y at point A of axially FG beam having square cross-section fixed-fixed (fx-fx) boundary condition for $m = 0, 0.5, 1, 2, 3$.

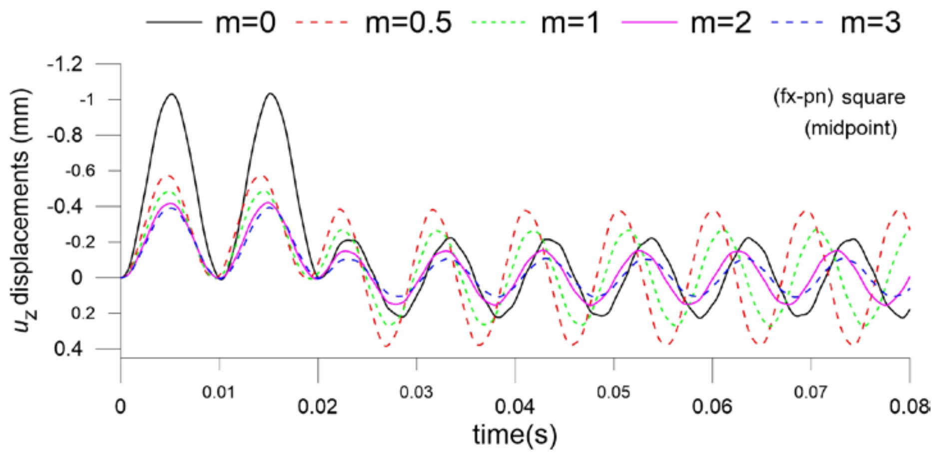

Figure 5(a). The vertical displacement u_z at midpoint of axially FG beam having square cross-section fixed-pinned (fx-pn) boundary condition for $m = 0, 0.5, 1, 2, 3$.

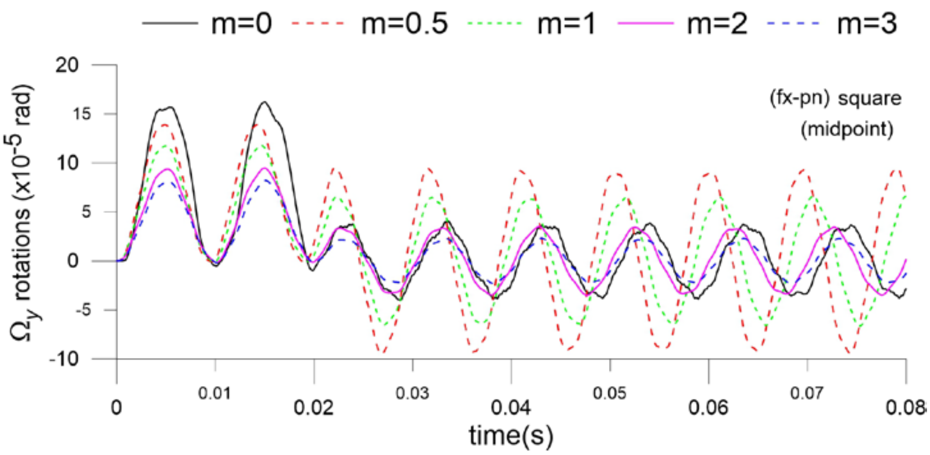

Figure 5(b). The rotation Ω_y at midpoint of axially FG beam having square cross-section fixed-pinned (fx-pn) boundary condition for $m = 0, 0.5, 1, 2, 3$.

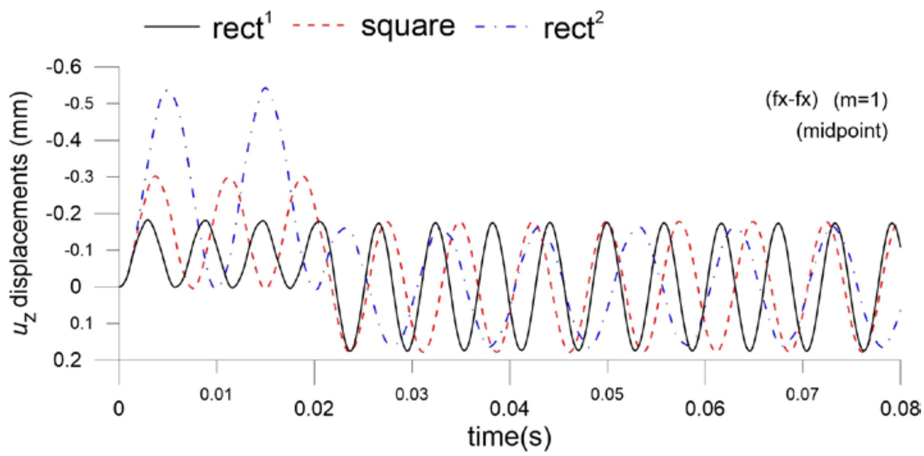

Figure 6(a). The vertical displacement u_z at midpoint of axially FG beam having the material gradient $m = 1$ fixed-fixed (fx-fx) boundary condition for rect^1, rect^2, square cross sections.

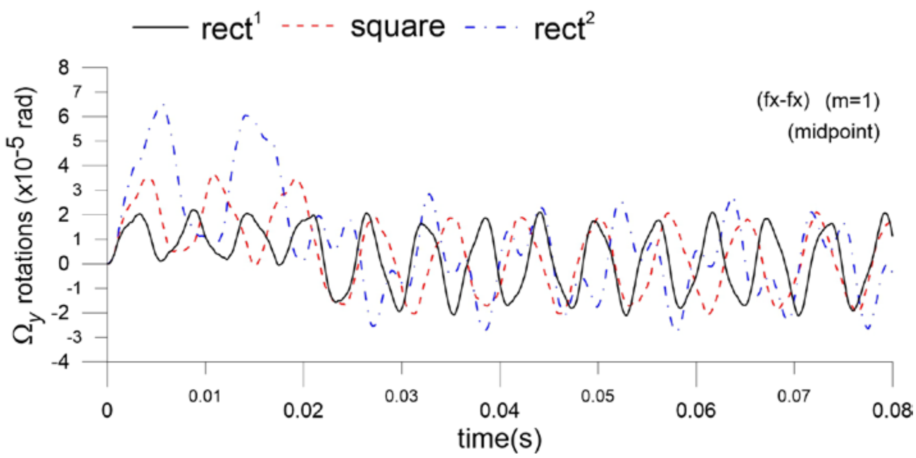

Figure 6(b). The rotation Ω_y at midpoint of axially FG beam having the material gradient $m = 1$ fixed-fixed (fx-fx) boundary condition for rect^1, rect^2, square cross sections.

Table 1. Percent reductions in the displacements and rotations for square cross-section in the case of $m = 1, 2, 3$ with respect to $m = 0.5$.

m	fixed-fixed		fixed-pinned	
	u_z (%)	Ω_y (%)	u_z (%)	Ω_y (%)
1	13.5	13.1	15.6	15.5
2	24.0	30.1	27.1	32.5
3	28.5	42.6	31.9	41.8

4. CONCLUSION

The forced vibration analysis of axially functionally graded (FG) straight beams under rectangular type impulsive load is investigated using the mixed finite element formulation based on the Timoshenko beam theory. The solutions are obtained in Laplace space and the results are transformed back to time space by using modified Durbin's algorithm. Some parametric studies are performed to observe the effect of the material gradient index, boundary condition and the type of cross section on the forced vibration analysis of the axially FG straight beam. Following remarks can be cited:

- The results of the forced vibration analysis performed by the mixed finite element method (MFEM) are in a well agreement with ANSYS.

- As the material gradient index ($m = 0.5, 1, 2, 3$) increases, a decreasing trend in the amplitude of u_z, and, Ω_y at the midpoint of the beam is observed for the forced vibration zone $(0 \leq t \leq t_{load} = 0.021\text{s})$. However, this conclusion is not extendable for the free vibration zone $(0.021 < t \leq 0.08\text{s})$ (see Figures 4(a-b), 5(a-b)).

- As expected, a reduction in the displacements is observed as the degree of indeterminacy of the structure increased (see Figures 4(a)-5(a)).

- An increase in the thickness of the cross-section caused a reduction of the displacement u_z and the rotation Ω_y at the midpoint of the beam (see Figures 6(a-b)).

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