INTERNATIONAL CIVIL ENGINEERING AND ARCHITECTURE CONFERENCE 2019
ICEARC'19
17-20 APRIL 2019 TRABZON / TURKEY
Karadeniz Technical University

FREE VIBRATION ANALYSIS OF AXIALLY FG STRAIGHT BEAMS BY MIXED FEM

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Abstract

The increased use of axially/transversely functionally graded beams in many applications due to their attractive properties in strength, stiffness and lightness has resulted in a growing demand for such as mechanical, civil, mechatronics and aerospace engineering in the design of structures (such as bridges, railways, aircrafts). The objective of this study is to investigate the free vibration analysis of axially functionally graded (FG) straight beams by using the mixed finite element formulation based on the Timoshenko beam theory. The functional is based on the Gâteaux differential and the potential operator concept. In the finite element formulation, two-noded straight element is used to discretize beam domain. In free vibration analysis, the problem of determining the natural frequencies of a structural system reduces to the solution of a standard eigenvalue problem. A condensation procedure is applied on the system matrix over the stress resultants. A convergence analysis for the natural frequencies of axially FG straight beams is carried out and the results are compared with a commercial structural software. In this study, the axially FG material is modelled by a power-law relation. A parametric study is performed in order to examine the effect of the material gradient on the natural frequencies of axially FG straight beams having different cross-sections and boundary conditions. The types of cross section are rectangular and square, and they have the same cross-sectional area. The boundary conditions are fixed-free, fixed-pinned and fixed-fixed.

Keywords: axially FG material, finite element, free vibration, straight beam, Timoshenko beam theory

1. Introduction

In virtue of their outstanding properties such as, strength, thermal characteristics, lightness *etc* axially/transversely functionally graded (FG) beams. are preferred in various applications with a growing demand for instance in mechanical, civil, mechatronics and aerospace engineering as structural members. In the literature, there are many studies about axially FG beams. Some studies can be cited as follows: Using semi-inverse method, Wu et.al. 2005 investigated the free vibration analysis of axially FG beams. Huang and Li 2010 studied natural frequencies of axially functionally graded beams with non-uniform cross section by reducing the governing equations to Fredholm integral equations. Longitudinal free vibration analysis of axially functionally graded microbars is investigated based on strain gradient elasticity theory by Akgöz and Civalek 2013. The free vibration analysis of axially exponentially graded beams is performed by Li et.al. 2013. Rajasekaran and Norouzzadeh Tochaei 2014 investigated the free vibration characteristics of axially FG tapered Timoshenko beams using differential transform element method (DTEM) and differential quadrature element method of the lowest-order (DQEL). Zeighampour and Beni 2015 examined the vibration of axially FG material (AFGM) nanobeams using the strain gradient theory. Zhao et.al. 2017 obtained the free vibration solution of axially FG Euler–Bernoulli and Timoshenko beams having non-uniform cross-sections using a new approach based on Chebyshev polynomials. Using the asymptotic development method (ADM), Cao et al. 2018 conducted free vibration analysis of axially FG beams. Implementing asymptotic perturbation approach (APA), Cao et.al. 2019 obtained a simple analytical expression for the free vibration analysis of non-uniform and non-homogenous beams with different boundary conditions. Zhou and Zhang 2019 presented an effective approach for uncertain natural frequency analysis of functionally graded beams with axially varying stochastic material properties.

In this study, free vibration characteristics of axially FG beams based on Timoshenko assumptions are investigated using a mixed finite element method (MFEM). A two-noded one dimensional element is adopted in the finite element formulation. The four degrees of freedom at each node are displacement, cross-sectional rotation, force and bending moment, respectively. A set of parametric analyses are introduced to reveal the influence of some parameters (e.g. cross-sections and boundary conditions of beam) on the natural frequencies of axially FG straight beams.

2. Formulation

2.1. Functional

The field equations and functional for the isotropic homogenous spatial beam based on Timoshenko beam theory exist in Omurtag and Aköz 1992 and Eratlı et.al. 2016. Letting xyz denotes the Cartesian coordinate system, $\rho = \rho(x)$ is the material density and x is the axis of the straight beam, the functional of an axially functionally graded straight beam can be given as follows:

$$\begin{aligned}\mathbf{I} = & -[u_z, T_{z,x}] + [\Omega_y, T_z] - [M_{y,x}, \Omega_y] - \frac{1}{2EI_y}[M_y, M_y] - \frac{k'}{2GA}[T_z, T_z] + \frac{1}{2}\rho A[\ddot{u}_z, \ddot{u}_z] + \frac{1}{2}\rho I_y[\ddot{\Omega}_y, \ddot{\Omega}_y] \\ & - [q_z, u_z] - [m_y, \Omega_y] + [(T_z - \hat{T}_z), u_z]_\sigma + [(M_y - \hat{M}_y), u_z]_\sigma + [\hat{u}_z, T_z]_\varepsilon + [\hat{\Omega}_y, M_y]_\varepsilon\end{aligned} \tag{1}$$

where u is the displacement, Ω is the rotation, T is the force, M is the moment, A is the cross sectional area, k' is the shear correction factor, I is the moment of inertia, E is the modulus of elasticity, G is the shear modulus, q_z and m_y are the distributed external force and moment. The parentheses in Eq. (1) indicate the inner product, and the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively. Once the motion is considered as harmonic for free vibration analysis, it is clear that, $q_z = m_y = 0$. Also, the acceleration terms can be written in the form $\omega^2 [u_z, u_z]$ and $\omega^2 [\Omega_y, \Omega_y]$, where ω is the natural circular frequency [see Eq.(1)]. The field equations and functional for a homogenous elastic spatial bar exists in Omurtag and Aköz 1992 and in this study it is reduced and revised for axially FG straight beams.

2.2 Mixed Finite Element Formulation

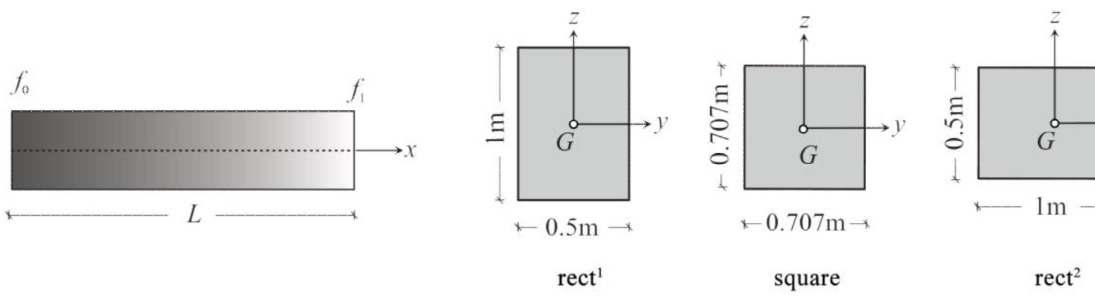
Two noded straight finite element is formulated by using linear shape functions. Nodal variables are vertical displacement, rotation of cross-section, vertical shear force, bending moment. Explicit form of the mixed finite element matrices of spatial bar exists in Omurtag and Aköz 1992.

2.3 Axially FG Material

Axially FG material distribution is assumed by a power-law relation along the axis of straight beam as follows:

$$f(x) = f_0 + (f_1 - f_0)\left(\frac{x}{L}\right)^m \tag{2}$$

where f denotes a material property (*e.g.* modulus of elasticity: E , density: ρ or modulus of shear: G), m is the material gradient index, the subscript "0" and "1" denotes the materials at left and right hand sides of the beam, respectively. x is the axis of beam and L is the length of the beam (Fig. 1.a).



(a) axially FG straight beam (b) cross-sections

Figure 1. Axially FG straight beam and cross sections

2.4 Dynamic Analysis

The determining the natural vibration frequencies of a structural system reduces to the solution of a standard eigenvalue problem,

$$\left([\mathbf{K}]-\omega^2[\mathbf{M}]\right)\mathbf{u}=\mathbf{0} \tag{3}$$

where $[\mathbf{K}]$ is the system matrix, $[\mathbf{M}]$ is the mass matrix for the entire domain, $\mathbf{u}$ is the eigenvector (mode shape) and ω denotes the natural angular frequency of the system. Hence the explicit form of standard eigenvalue problem in the mixed formulation is

$$\left(\begin{bmatrix}[\mathbf{K}_{11}] & [\mathbf{K}_{12}]\\ [\mathbf{K}_{22}] & [\mathbf{K}_{22}]\end{bmatrix}-\omega^2\begin{bmatrix}[\mathbf{0}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{M}]\end{bmatrix}\right)\begin{Bmatrix}\begin{Bmatrix}\mathbf{T}_z\\ \mathbf{M}_y\end{Bmatrix}\\ \begin{Bmatrix}\mathbf{\Omega}_y\\ \mathbf{u}_z\end{Bmatrix}\end{Bmatrix}=\begin{Bmatrix}\{\mathbf{0}\}\\ \{\mathbf{0}\}\end{Bmatrix} \tag{4}$$

where $\{\mathbf{F}\}=\{\mathbf{T}_z \quad \mathbf{M}_y\}^T$ denotes the nodal force and the moment vectors and $\{\mathbf{U}\}=\{\mathbf{\Omega}_y \quad \mathbf{u}_z\}^T$ signifies the nodal displacement and rotation vectors. To attain consistency between Eqs. (3) and (4), the $\{\mathbf{F}\}$ vector is eliminated in Eq. (4), which yields to the condensed system matrix $[\mathbf{K}^*]=[\mathbf{K}_{22}]-[\mathbf{K}_{12}]^T[\mathbf{K}_{11}]^{-1}[\mathbf{K}_{12}]$. Finally, the eigenvalue problem in the mixed formulation becomes,

$$\left([\mathbf{K}^*]-\omega^2[\mathbf{M}]\right)\mathbf{U}=\mathbf{0} \tag{5}$$

3. Numerical Examples

Some parametric analyses are performed for an axially functionally graded (AFG) straight beam over two different rectangular and a square cross-section and various boundary conditions (fixed-fixed, fixed-pinned, fixed-free). The influence of these parameters on the natural frequency of the AFG straight beam is investigated under the variation of the material gradient index.

Example 1: An axially FG straight beam having fixed-free boundary condition given in Zhao 2017 is solved in order to show the performance of the used mixed FE formulation and then the results are compared with each other. The material and geometrical properties for the axially FG straight beam are: $E_0=200\text{GPa}$, $\rho_0=5700\,\text{kg/m}^3$, $E_1=70\text{GPa}$, $\rho_1=2702\,\text{kg/m}^3$ and the Poisson's ratio $\upsilon=0.3$. The material gradient $m=1$ in Eq.(2). The dimensionless radius of gyration is $r=\sqrt{I/AL^2}$ and the dimensionless natural frequency is $\Omega=\omega\sqrt{\rho_0 AL^4/E_0 I}$. The dimensionless fundamental natural frequency of axially FG beam is 3.896370 in Zhao 2017 and 3.89613 in mixed FEM for 50 elements, respectively. The dimensionless fundamental natural frequency of mixed FE result is compared with the result given in Zhao 2017 and the percent difference is -0.01%.

Example 2: The objective of this example is to investigate the effect of material gradients on the natural frequencies of axially FG straight beam for different cross sections (square and rectangular) where all the cross-sections are chosen to have the same constant area and boundary conditions (fixed-fixed, fixed-free, fixed-pinned). The geometrical parameters of FG straight beam: the length of beam is $L=5\text{m}$, the dimensions of the rectangular cross-sections for rect[1] and rect[2], and dimension of square cross-section are given in Fig 1.b. The material properties are $E_0=210\text{GPa}$, $\rho_0=7500\,\text{kg/m}^3$, $E_1=70\text{GPa}$, $\rho_1=2500\,\text{kg/m}^3$, the Poisson's ratio $\upsilon=0.3$. The material gradients are $m=0.5,1,2,3$ in Eq. (2). The fundamental natural frequency of axially FG straight beam with $m=3$, having a square cross-section and fixed-free boundary condition is obtained by using ANSYS (2450 SOLID186 elements). Axially FG material is defined by the average value of material properties between each node of the straight beam (51 nodes). The percent difference of the fundamental natural frequencies of ANSYS with respect to MFEM (50 elements) is −0.57%. The out-of-plane natural frequencies of axially FG straight beam are considered and the first three natural frequencies are given in Tables 1-3 for 50 elements. Below, the following abbreviation "BC" is used for boundary condition.

The effect of boundary condition: The fundamental natural frequency values of beams having fixed-pinned and fixed-free BCs are normalized with respect to beams having fixed-fixed BC that correspond rect[1], square and rect[2] cross sections and the percent decreases in the natural frequencies are given in Table 4 for all material gradient

$m = 0.5,1,2,3$. It is observed that the maximum percent reduction is obtained for rect[2], $m = 0.5$ and fixed-free boundary condition, the minimum percent reduction is obtained for rect[1], $m = 3$ and fixed-pinned boundary condition (see Table 4).

***The effect of cross-sections*:** The fundamental natural frequency values of the beams having square and rect[2] cross-sections are normalized with respect to the results of beams having rect[1] cross-section for all boundary conditions. The percent decreases in the natural frequencies are given in Table 5 for all material gradient indexes $m = 0.5,1,2,3$. It is observed that the percent reductions in the natural frequencies are not influenced from the change of material gradient index (see Table 5).

***The effect of material gradient index*:** The fundamental natural frequency values of $m = 1,2,3$ are normalized with respect to $m = 0.5$ that correspond rect[1], square and rect[2] cross sections and the percent differences in the natural frequencies are given for fixed-fixed and fixed-pinned boundary conditions in Fig.2, and for fixed-free boundary condition in Table 6. As the material gradient index increases, a decreasing trend in the fundamental natural frequencies is observed for fixed-fixed and fixed-pinned boundary conditions (see Fig.2), however there is no specific trend for fixed-free boundary condition (see Table 6).

Table 1. The natural frequencies (in Hz) of axially straight beams having fixed-fixed BC.

BC			material gradient index (m)			
	Cross-section	Mode No	0.5	1	2	3
fixed-fixed	rect[1]	1	177.5	171.2	164.1	161.3
		2	404.2	395.6	385.7	381.4
		3	671.9	662.8	653.2	648.4
	square	1	138.3	133.3	127.8	125.7
		2	335.5	328.4	320.6	317.2
		3	580.6	572.8	564.6	560.5
	rect[2]	1	103.6	99.8	95.7	94.1
		2	263.8	258.3	252.4	249.8
		3	475.4	469.1	462.5	459.3

Table 2. The natural frequencies (in Hz) of axially straight beams having fixed-pinned BC.

BC			material gradient index (m)			
	Cross-section	Mode No	0.5	1	2	3
fixed-pinned	rect[1]	1	140.7	138.0	133.6	131.6
		2	368.5	363.4	357.4	355.0
		3	643.5	637.6	632.0	629.6
	square	1	106.1	104.0	100.8	99.4
		2	295.8	291.9	287.6	285.8
		3	542.0	537.3	533.0	531.3
	rect[2]	1	77.7	76.2	73.9	72.9
		2	226.6	223.7	220.6	219.4
		3	433.3	429.7	426.5	425.4

Table 3. The natural frequencies (in Hz) of axially straight beams having fixed-free BC.

BC			material gradient index (m)			
	Cross-section	Mode No	0.5	1	2	3
fixed-free	rect[1]	1	43.9	45.0	43.5	41.7
		2	196.9	198.0	197.3	196.4
		3	442.0	442.0	443.1	444.9
	square	1	31.6	32.4	31.3	30.0
		2	151.1	152.0	151.7	151.2
		3	359.2	359.2	360.4	361.9
	rect[2]	1	22.5	23.1	22.3	21.4
		2	112.0	112.7	112.6	112.3
		3	278.1	278.1	279.2	280.5

Table 4. The percent reductions in the fundamental natural frequencies of axially FG beams in the case of fixed-pinned and fixed-free BCs with respect to fixed-fixed BC.

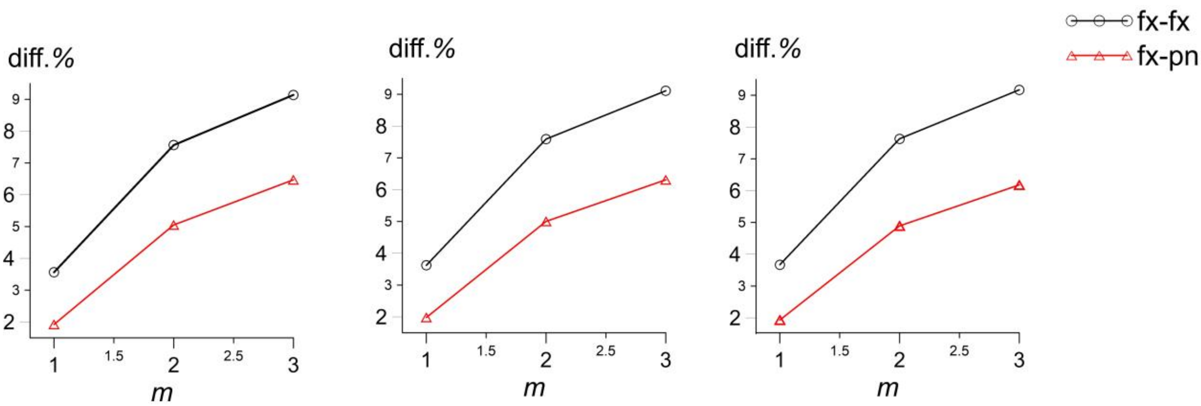
		material gradient index (m)			
BC	Cross-section	0.5	1	2	3
fixed-pinned	rect[1]	20.7%	19.4%	18.6%	18.4%
	square	23.3%	22.0%	21.1%	20.9%
	rect[2]	25.0%	23.6%	22.8%	22.5%
fixed-free	rect[1]	75.3%	73.7%	73.5%	74.1%
	square	77.2%	75.7%	75.5%	76.1%
	rect[2]	78.3%	76.9%	76.7%	77.3%

Table 5. The percent reductions in the fundamental natural frequencies of axially FG beams in the case of square and rect[2] cross-sections with respect to rect[1] cross-section.

		material gradient index (m)			
Cross-section	BC	0.5	1	2	3
square	fixed-fixed	22.1%	22.1%	22.1%	22.1%
	fixed-pinned	24.6%	24.6%	24.6%	24.5%
	fixed-free	28.0%	28.0%	28.0%	28.1%
rect[2]	fixed-fixed	41.6%	41.7%	41.7%	41.7%
	fixed-pinned	44.8%	44.8%	44.7%	44.6%
	fixed-free	48.7%	48.7%	48.7%	48.7%

Table 6. The percent differences in the natural frequencies of axially FG beams in the case of $m = 1,2,3$ with respect to $m = 0.5$.

BC			material gradient index (m)		
	Cross-section	Mode No	1	2	3
fixed-free	rect[1]	1	-2.51%	0.91%	5.01%
		2	-0.56%	-0.20%	0.25%
		3	0.00%	-0.25%	-0.66%
	square	1	-2.53%	0.95%	5.06%
		2	-0.60%	-0.40%	-0.07%
		3	0.00%	-0.33%	-0.75%
	rect[2]	1	-2.67%	0.89%	4.89%
		2	-0.63%	-0.54%	-0.27%
		3	-0.04%	-0.43%	-0.90%



(a) rect[1] (b) square (c) rect[2]

Figure 2. The percent differences in the natural frequencies of axially FG beam. The gradient index results of $m = 1,2,3$ are normalized with respect to the result of $m = 0.5$. fx-fx: fixed-fixed BC, fx-pn: fixed pinned BC.

4.Conclusion

The free vibration analysis of axially functionally graded straight beams is investigated using a straight mixed finite element formulation based on the Timoshenko beam theory. The mixed finite element formulation is verified with the example given in Zhao 2017 (see Ex.1) and the commercial program ANSYS (Ex.2) for an axially functionally graded straight beam. Some parametric analysis is performed over material gradient index in order to observe the effect of the boundary conditions, different rectangular cross-sections and square cross-section (keeping the area constant) on the dynamic behaviour of axially functionally graded straight beam. Following remarks can be cited:

- As the degree of indeterminacy of straight beam increases, an increasing trend in the natural frequencies is observed.
- The natural frequencies of the out-of-plane vibration decrease when the height of cross section is less than the width of it (keeping the cross-sectional area constant).
- The fundamental natural frequencies are directly influenced from the boundary conditions.
- Although a decreasing trend in the fundamental natural frequencies with respect to increasing material gradient index is figured out under fixed-fixed and fixed-pinned boundary conditions, it is not possible to extend this conclusion for fixed-free boundary condition.

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